

# Magnetostatic field noise near metallic surfaces

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**Abstract.** We develop an effective low-frequency theory of the electromagnetic field in equilibrium with thermal objects. The aim is to compute thermal magnetic noise spectra close to metallic microstructures. We focus on the limit where the material response is characterised by the electric conductivity. At the boundary between empty space and metallic microstructures, a large jump occurs in the dielectric function which leads to a partial screening of low-frequency magnetic fields generated by thermal current fluctuations. We resolve a discrepancy between two approaches used in the past to compute magnetic field noise spectra close to microstructured materials.

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## 1 Introduction

In the context of atom chips, low-frequency thermal magnetic noise has recently emerged as one crucial element that limits the lifetime of miniaturised atom traps. Recent experiments [1–3] have confirmed the basic features of theoretical predictions for spin-flip processes induced by magnetic-near-field fluctuations. A current trend is to extract other physical mechanisms that lead to loss by subtracting the near-field induced loss rate. One particularly interesting mechanism is the lowering of the trap depth due to atom-surface interactions [3]. Accurate calculations of magnetic near-field noise are clearly needed for this purpose. Magnetic fluctuations are also relevant in other contexts, for example in biophysics where they impose ultimate limits on the sensitivity of SQUID detectors [4], and in magnetic resonance force microscopy, a near-field variant of magnetic resonance imaging [5,6].

Typically, one is interested in field frequencies  $\hbar\omega \ll k_B T$  where the noise is dominantly classical. The border of the quantum regime can be reached with magnetically trapped atoms, either by cooling the microtrap components and/or applying strong static magnetic fields that push up the relevant frequency range (given by the Larmor frequency). Even at the highest frequencies conceivable with state-of-the-art atom chip structures (in the GHz range), the (vacuum) field wavelength  $\lambda$  is much larger than the characteristic distances, so that the quasi-static approximation applies outside the structures. This leads to a peculiar situation to describe the field fluctuations: one cannot apply the standard procedure and attribute

thermal or quantum fluctuations to the normal mode amplitudes of the field, because there are no nontrivial solutions to the homogeneous field equations (i.e., eigenmodes) in the quasi-static limit. Near-field noise is actually dominated by the fluctuations of its sources (currents, magnetic moments) whose spectral mode density depends on material or atomic constants [4,7–10]. As a result, the near-field noise spectrum differs markedly from the celebrated blackbody radiation law [4,11,12].

Roughly, two approaches can be identified to compute magnetic noise close to micro- and nanostructures. The first one can be traced back to the fluctuation electrodynamics put forward by Rytov and co-workers [13] in the 1950's. Based on a statistical thermodynamics argument (the fluctuation-dissipation — FD — theorem [14]), random charge and current fluctuations are associated to a dissipative material structure. Their radiation is incoherently summed to give the total noise strength of the field. In a planar geometry, the radiated field and the required averaging can be calculated analytically. Results along this line have been computed and experimentally verified for planar metallic layers by Varpula and Poutanen in 1984 [4]. Sidles and co-workers give an extensive discussion with applications for magnetic resonance microscopy and quantum computing [6].

An alternative approach uses the FD theorem for the (magnetic) field itself and has been popularised in a series of papers by Agarwal in 1975 [15]. The advantage is that the incoherent averaging is avoided; the FD theorem reduces the calculation to the radiation of a single dipole source (Green function), located at the observation point. This method has been applied, in the context of atomic

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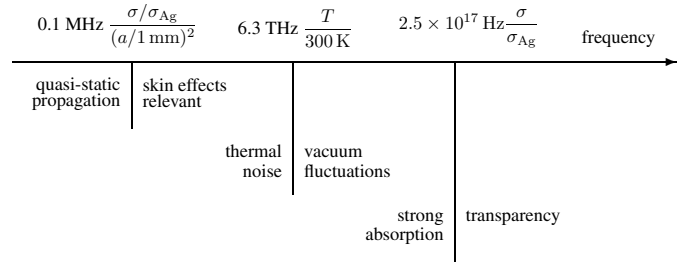
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microtraps, by the present author and co-workers [16] and Rekdal and co-workers [17, 18]. Both approaches have been shown to be equivalent under fairly general conditions, thanks to an identity that implements the FD theorem in electromagnetism [19, 20].

In the context of integrated atom optics, incoherent summation over fields has been put forward by Pötting and the present author as a versatile tool to handle arbitrary nanostructures [21]. It just remains to perform a certain spatial integral over the volume filled with electrically conducting material. This yields the correct scaling of the noise spectrum with the atom chip geometry, provided the skin depth is long enough. Based on this approach, for example, the Vuletic group could describe spin-flip loss rates close to rectangular wires reasonably well [3] (see also Refs. [18, 22] for a re-analysis and discussion). A closer comparison shows, however, that the theoretical results are off by numerical factors between two and three compared to the noise spectrum predicted by the FD theorem [21]. This discrepancy is the motivation for the present paper. We point out an error in the ‘incoherent summation’ approach that is linked to the particular boundary conditions for the electromagnetic field at the surface of a good conductor. We derive approximate boundary conditions that apply to any geometry in the low-frequency range relevant for atom chips. In the planar case, we show that they lead to an accurate agreement with the ‘Green function’ approach in the limit that the vacuum wavelength is the largest length scale. The only point missing in the theory is the blackbody noise level that prevails at large distances, but this one is in most situations impossible to detect anyway.

The parameter regime we focus on in this paper is illustrated in Figure 1. We shall call ‘quasi-static’ the regime where the skin depth  $\delta$  inside the material is larger than any other geometrical scale (denoted  $a$  in Fig. 1). We focus on metallic materials and use the definition  $\delta = (\mu_0\sigma\omega/2)^{-1/2}$  in terms of the (DC) conductivity  $\sigma$ . Our theory aims at covering both the quasi-static regime and a skin depth comparable to  $a$ . The temperature  $T$  defines another frequency scale below which the field fluctuations behave classically. This is actually not a limitation as long as we assume thermal equilibrium. The theory is extended into the quantum regime with the replacement  $k_B T \mapsto \frac{1}{2}\hbar\omega \coth(\hbar\omega/2k_B T)$  and assuming symmetrised noise correlation spectra. At frequencies in the visible and ultraviolet range, however, the dielectric function of the material becomes complicated, and more parameters (transverse optical phonon frequency, plasma frequency, tabulated data...) are needed for an accurate modelling.

The present paper thus aims at clarifying the validity of the ‘incoherent summation’ approach and at extending it into the regime of a short skin depth using the appropriate boundary conditions. We also argue that from a practical point of view, the FD approach appears simpler because it is sufficient to compute the radiation from a single point source, while for ‘incoherent summation’, many sources (anywhere inside the spatial domains filled with absorbing material) have to be treated.



**Fig. 1.** Characteristic frequencies involved in near-field electromagnetic noise. The conductivity of silver,  $\sigma_{\text{Ag}}$  is used as a convenient scaling parameter. A typical geometrical feature size is denoted by  $a$ . Three characteristic frequencies separate different regimes as illustrated by the vertical lines. The formulas at the frequency axis give explicit values and their scaling with the relevant parameters. For example, on a scale  $a = 10 \mu\text{m}$ , the skin effect is irrelevant for the field propagation near a silver structure at frequencies below  $0.1 \text{ MHz} \times 10^4 = 10^3 \text{ MHz}$ ; the quasi-static approximation applies in this regime. At frequencies approaching  $2.5 \times 10^{17} \text{ Hz}$ , silver becomes transparent, and its permittivity no longer involves a purely real conductivity.

In the following sections, we start by writing down the basic equations for the magnetic field and explain the relevant parameter regimes (Sect. 2). The boundary condition at the surface of a good conductor is derived. In Section 3, we review the incoherent summation technique and show for the special case of a metallic half-space that with the correct boundary condition, one gets a magnetic noise spectrum in agreement with the FD approach. We show that in the limit of a short skin depth, the transmission of the magnetic field out of the metal becomes much less efficient. In Section 3.4, we give a qualitative explanation of the power laws in the distance-dependence of the noise spectrum and review results obtained for a thin metallic layer. In Section 4, we formulate the equations to be solved within the FD approach when the quasi-static approximation is made in the spatial domains filled with vacuum. The formulation applies to an arbitrary geometry and is then specialised to a metallic half-space. In the latter case, we demonstrate agreement with the more complex, fully retarded FD approach in the long vacuum wavelength limit.

## 2 Boundary conditions at low frequency

We want to solve the Maxwell equations in a non-magnetic medium ( $\mu(\mathbf{x}) \equiv \mu_0$ ):

$$\nabla \cdot (\sigma \mathbf{E} + \mathbf{j}) = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad (1)$$

$$\nabla \times \mathbf{E} = i\omega \mathbf{B}, \quad \nabla \times \mathbf{B} = \mu_0 (\sigma \mathbf{E} + \mathbf{j} + \nabla \times \mathbf{M}) \quad (2)$$

where  $\sigma(\mathbf{x})$  is the metal conductivity. In the vacuum regions,  $\sigma(\mathbf{x}) = -i\epsilon_0\omega$  which will be assumed much smaller in magnitude than the metal conductivity, denoted  $\sigma$ . Two kinds of sources appear here that apply to the methods of magnetic noise calculations mentioned in the introduction.

They are represented by the externally prescribed terms  $\mathbf{j}(\mathbf{x})$  and  $\mathbf{M}(\mathbf{x})$ . In the incoherent summation technique, the current density  $\mathbf{j}(\mathbf{x})$  is localised inside the metal and represents thermal fluctuations. It is a random quantity with correlation function given by equation (14). For the calculation of the magnetic Green function, the magnetisation  $\mathbf{M}(\mathbf{x})$  can be identified with the magnetic moment of an atom localised in vacuum outside the metal, see after equation (37). Depending on the method of calculation, only one or the other source term is actually nonzero.

Taking the curl of (2b), we find the following wave equation for the magnetic field

$$\nabla^2 \mathbf{B} + q^2 \mathbf{B} = -\mu_0 [\nabla \times \mathbf{j}(\mathbf{x}) - \nabla \times \nabla \times \mathbf{M} + (\nabla \sigma) \times \mathbf{E}]. \quad (3)$$

Outside the metal,  $q(\mathbf{x}) = \omega/c$ . Inside the metal,  $q(\mathbf{x})$  is complex and related to the skin depth  $\delta$  by  $q^2 = 2i/\delta^2 = i\mu_0\sigma\omega$ . We could also introduce a spatial diffusion coefficient  $D = \mu_0\sigma$  by going back to time-dependent equations. The source terms of the magnetic wave equation (3) are related to the current density, the magnetisation density and to a current parallel to the surface induced by the electric field. It turns out that this last current leads to a jump in the derivative of the magnetic field across the interface. The solutions to the wave equation can be sought in terms of a Green function, assuming localised sources. This is why we assume in the following that the current  $\mathbf{j}(\mathbf{x})$  is nonzero only inside the metal.

## Boundary conditions

The magnetic field itself is continuous across the interface of the metal (assumed to be non-magnetic). The electric field components parallel to the interface are continuous as well. For the normal components, the continuity of the displacement field gives

$$\frac{2i}{\delta^2} \mathbf{n} \cdot \mathbf{E}|_{\text{in}} = \frac{(2\pi)^2}{\lambda^2} \mathbf{n} \cdot \mathbf{E}|_{\text{out}}. \quad (4)$$

We shall assume that the wavelength  $\lambda$  is much larger than any other relevant length scales in the problem, and take the limit  $\delta \ll \lambda \rightarrow \infty$ . To lowest order, this transforms (Eq. (4)) into

$$\mathbf{n} \cdot \mathbf{E}|_{\text{in}} = 0. \quad (5)$$

We check explicitly below that this boundary condition yields results consistent with a fully retarded calculation.

We next calculate the jump condition for the magnetic field due to the surface current. Assume first a planar metallic surface located at  $z = 0$  with unit outward normal  $\mathbf{n}$ . Noting that  $\nabla\sigma(\mathbf{r}) = -\sigma\mathbf{n}\delta(z)$  and integrating the magnetic wave equation along a path perpendicular to the interface, we find the following jump condition

$$\left. \frac{\partial}{\partial n} \mathbf{B} \right|_{\text{in}}^{\text{out}} = \mu_0 \sigma \mathbf{n} \times \mathbf{E}. \quad (6)$$

where the scripts ‘in’ and ‘out’ mark the field inside and outside the metal. The electric field is taken at the interface because only its tangential components are involved.

We can also get this boundary condition directly from the Maxwell equations and without specifying a planar boundary. At the metal surface, we evaluate (2b) just above and below the interface and find, using that the current  $\mathbf{j}$  vanishes at the interface,

$$\nabla \times \mathbf{B}|_{\text{in}}^{\text{out}} = \mu_0 \sigma \mathbf{E}|_{\text{in}}^{\text{out}} = -\mu_0 \sigma \mathbf{E}|_{\text{in}} \quad (7)$$

The last equality is again due to the negligible vacuum conductivity. (Note that  $\varepsilon_0\omega/\sigma = (2\pi\delta/\lambda)^2/2 \rightarrow 0$  in lowest order.) Taking components parallel to the interface, we find

$$\mathbf{n} \times (\nabla \times \mathbf{B})|_{\text{in}}^{\text{out}} = -\mu_0 \sigma \mathbf{n} \times \mathbf{E}. \quad (8)$$

Now, from  $\nabla \cdot \mathbf{B} = 0 = \partial_k (\nabla \cdot \mathbf{B}) = \partial_i \partial_k B_i$ , we can derive using the Gauss theorem:  $n_i \nabla B_i|_{\text{in}}^{\text{out}} = 0$ . This identity cancels one of the terms coming from the expansion of the double vector product in (8). We thus find the jump condition (6). Note that the present derivation is valid for any geometry of the interface — which is less obvious for the previous one because one has to integrate the Laplacian operator.

## 3 Incoherent summation

In this section, we focus on a planar metallic surface and the quasi-static limit (geometrical distances even smaller than the skin depth). We display the surface electric field and the correction it implies for the transmitted magnetic field. Technical details are deferred to Appendix A.

### 3.1 Transmitted field expansion

For a planar surface parallel to the  $xy$ -plane, an expansion in plane waves with two-dimensional wave vectors is straightforward. We use the notation  $\mathbf{K} = (k_x, k_y, 0)$  and find just below the metal surface the following expansion

$$\mathbf{B}(\mathbf{r}) = \int \frac{d^2 K}{(2\pi)^2} (\mathbf{B}_i[\mathbf{K}] e^{i\mathbf{k}_i \cdot \mathbf{r}} + \mathbf{B}_r[\mathbf{K}] e^{i\mathbf{k}_r \cdot \mathbf{r}}) \quad (9)$$

where  $\mathbf{k}_{i,r} = \mathbf{K} \pm i\kappa\mathbf{n}$  with  $\kappa = \sqrt{K^2 - q^2}$  ( $\text{Re } \kappa > 0$ ). In the quasi-static limit, we have  $K \gg |q|$ , and therefore  $\kappa \approx K$ . More general formulas can be found in Appendix A. As shown there, a current density localised below the metal surface produces an ‘incident’ magnetic field with Fourier transform

$$\mathbf{B}_i[\mathbf{K}] = \frac{i\mu_0}{2K} \mathbf{k}_i \times \mathbf{J}[\mathbf{K}] \quad (10)$$

where

$$\mathbf{J}[\mathbf{K}] = \int_{-\infty}^0 dx_3 e^{Kx_3} \mathbf{j}[\mathbf{K}; x_3] \quad (11)$$

with  $\mathbf{j}[\mathbf{K}; x_3]$  being the 2D spatial Fourier transform of  $\mathbf{j}(\mathbf{x})$ . For the solution of the reflection/transmission

problem, we also need the electric field at the interface. Its tangential components are given by

$$\mathbf{n} \times \mathbf{E}[\mathbf{K}] = -\frac{\mathbf{n} \times \mathbf{K}}{\sigma K} (\mathbf{k}_i \cdot \mathbf{J}[\mathbf{K}]) = 2\mathbf{n} \times \mathbf{E}_i[\mathbf{K}]. \quad (12)$$

The factor of two between the ‘incident field’ and the actual field at the interface, may be explained intuitively by working with image currents (or dipoles): to fulfill the boundary conditions, one combines the fields of the actual dipole (inside the metal) and of an image dipole. Since the field is incident from a region with a large ‘refractive index’, the reflected field has the opposite sign: source and image dipoles therefore have the same polarity if they are parallel to the interface. As a consequence, their field components parallel to the interface double. Dipoles perpendicular to the interface have mirror images with the opposite polarity, leading to the cancellation of the normal field component, as required by the boundary condition (5).

Combining the jump condition (6) for the magnetic field with the in-plane electric field (12), we get the transmitted magnetic field:

$$\mathbf{B}_t[\mathbf{K}] = \frac{i\mu_0}{2K} \mathbf{k}_t \times \mathbf{J}[\mathbf{K}] + \frac{\mu_0}{2K^2} (\mathbf{n} \times \mathbf{K}) (\mathbf{k}_t \cdot \mathbf{J}[\mathbf{K}]) \quad (13)$$

where  $\mathbf{k}_t = \mathbf{K} + iK\mathbf{n}$ . With respect to the incident field (10), we thus have an additional term with components parallel to the interface. This term is absent when computing the field generated by the current density  $\mathbf{j}$  as if the latter were located in vacuum. It ensures in particular that the transmitted field vanishes if  $\mathbf{j}$  is parallel to  $\mathbf{n}$ , which is a well-known result for a planar geometry (see, e.g., [4]). We show now that the interference between both terms reduces the field fluctuations for some components, and reproduces the noise tensor found asymptotically from a retarded calculation.

### 3.2 Magnetic field correlation tensor

We first compute the correlation function of the Fourier transformed current. Local thermodynamic equilibrium gives the basic relation [13, 23] for the current fluctuations

$$\langle j_k^*(\mathbf{x}_1; \omega) j_l(\mathbf{x}_2; \omega') \rangle = 2\pi S(\mathbf{x}_1; \omega) \delta_{kl} \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(\omega - \omega'), \quad (14)$$

and we find

$$\langle J_k^*[\mathbf{K}_1; \omega] J_l[\mathbf{K}_2; \omega'] \rangle = \frac{(2\pi)^3}{2K_1} S(\omega) \delta_{kl} \delta(\mathbf{K}_1 - \mathbf{K}_2) \delta(\omega - \omega'). \quad (15)$$

The noise spectrum behaves like  $S(\mathbf{x}; \omega) \approx 2\sigma(\mathbf{x}) k_B T$  at low (sub-thermal) frequencies. The magnetic noise tensor  $\mathcal{B}_{ij}(\mathbf{x}_1, \mathbf{x}_2)$  is defined by

$$\langle B_i^*(\mathbf{x}_1; \omega) B_j(\mathbf{x}_2; \omega') \rangle = 2\pi \delta(\omega - \omega') \mathcal{B}_{ij}(\mathbf{x}_1, \mathbf{x}_2). \quad (16)$$

Whenever no confusion is possible, we suppress the frequency dependence for simplicity. At low frequencies, the magnetic noise spectrum tends towards a constant anyway.

The spatial Fourier transformed magnetic field (13) thus yields, using the current correlation function (15),

$$\mathcal{B}_{ij}(\mathbf{x}_1, \mathbf{x}_2) = \frac{\mu_0^2 S}{4} \int \frac{d^2 K}{2(2\pi)^2 K^5} e^{i\mathbf{K} \cdot (\mathbf{x}_2 - \mathbf{x}_1) - K(z_1 + z_2)} X_{ij} \quad (17)$$

$$X_{ij} = (-iK \epsilon_{ikl} k_k^* + (\mathbf{n} \times \mathbf{K})_i k_l^*) \delta_{lq} \\ \times (iK \epsilon_{j pq} k_p + (\mathbf{n} \times \mathbf{K})_j k_q). \quad (18)$$

This tensor can be worked out using the relations  $\mathbf{k} \times \mathbf{k}^* = (\mathbf{K} + iK\mathbf{n}) \times (\mathbf{K} - iK\mathbf{n}) = 2iK\mathbf{n} \times \mathbf{K}$  and  $|\mathbf{k}|^2 = 2K^2$ , valid in the quasi-static limit. We find

$$X_{ij} = K^2 (2K^2 \delta_{ij} - k_i k_j^* - 2(\mathbf{n} \times \mathbf{K})_i (\mathbf{n} \times \mathbf{K})_j) \quad (19)$$

where the last term includes also the crossed correlations between the two terms in equation (13), leading to the minus sign. For the planar geometry, we can work out the integral over the azimuthal angle. For simplicity, we focus on the noise tensor at the same position  $\mathbf{r} = \mathbf{x}_1 = \mathbf{x}_2$ . The angular average (denoted by double brackets) gives

$$\langle\langle k_i k_j^* \rangle\rangle = \frac{1}{2} K^2 \Delta_{ij} + K^2 n_i n_j \quad (20)$$

$$\langle\langle (\mathbf{n} \times \mathbf{K})_i (\mathbf{n} \times \mathbf{K})_j \rangle\rangle = \frac{1}{2} K^2 \Delta_{ij} \quad (21)$$

$$\Rightarrow \langle\langle X_{ij} \rangle\rangle = K^4 \left( \frac{1}{2} \Delta_{ij} + n_i n_j \right) \equiv K^4 s_{ij} \quad (22)$$

where  $\Delta_{ij} = \text{diag}(1, 1, 0)$  is the in-plane Kronecker symbol and  $s_{ij} = \text{diag}(\frac{1}{2}, \frac{1}{2}, 1)$  an anisotropic tensor that was also found in reference [16], using the asymptotic expansion of the fully retarded magnetic noise tensor. Thanks to the additional term in (19), we thus find the correct magnetic correlation tensor. Without this term,  $\text{diag}(\frac{3}{2}, \frac{3}{2}, 1)$  would have come out. Let us check the prefactor of the spectrum. It is given by

$$\mathcal{B}_{ij}(\mathbf{r}; \omega) = \frac{\mu_0^2 S}{4} \int_0^\infty \frac{dK}{4\pi K^4} e^{-2Kz} \langle\langle X_{ij} \rangle\rangle \quad (23) \\ = \frac{\mu_0^2 S}{32\pi z} s_{ij} = \frac{\mu_0^2 \sigma k_B T}{16\pi z} s_{ij}.$$

This is the result given in equation (24) of reference [16], taken in the quasi-static limit (distance  $z$  small compared to the skin depth). We thus have shown that when the correct boundary conditions for the magnetic field are used, the ‘incoherent summation’ approach is equivalent to the more rigorous FD theorem.

### 3.3 Impact of finite skin depth

We now discuss what the previous formulas become when the relevant geometrical distances are comparable to the skin depth  $\delta$ . The transmitted magnetic field takes the form (see Appendix A)

$$\mathbf{B}_t[\mathbf{K}] = \frac{\mu_0}{\kappa + K} \mathbf{k}_t \left( (\mathbf{n} \times \hat{\mathbf{K}}) \cdot \mathbf{J}[\mathbf{K}] \right). \quad (24)$$

Here,  $\hat{\mathbf{K}}$  is the unit vector along  $\mathbf{K}$ . This result is transverse, as it should be, because the vacuum wave vector satisfies  $\mathbf{k}_t^2 = 0$ . It also coincides with equation (13) in the quasi-static limit, as a simple calculation shows. Note that, again,  $\mathbf{B}_t = \mathbf{0}$  if  $\mathbf{j} \parallel \mathbf{n}$ .

The magnetic correlation tensor obtained from equation (24) can be worked out and reduces to the following simple formula

$$\mathcal{B}_{ij}(\mathbf{r}; \omega) = \frac{\mu_0^2 \sigma k_B T}{2\pi} s_{ij} \int_0^\infty \frac{K^3 dK e^{-2Kz}}{\text{Re } \kappa |\kappa + K|^2}. \quad (25)$$

It is interesting to note that the anisotropy tensor  $s_{ij} = \text{diag}(\frac{1}{2}, \frac{1}{2}, 1)$  describes the magnetic noise through the entire distance range. The asymptotic limits for the integral are

$$\int_0^\infty \frac{K^3 dK e^{-2Kz}}{\text{Re } \kappa |\kappa + K|^2} \approx \begin{cases} \frac{1}{8z}, & z \ll \delta \\ \frac{3\delta^3}{16z^4}, & z \gg \delta. \end{cases} \quad (26)$$

The second limit reproduces the result obtained asymptotically from the exact solution in the skin-dominated regime.

Let us analyse this skin-dominated limit in more detail and consider here the case  $|q| = \mathcal{O}(1/\delta) \gg K$ . We then have  $\kappa \approx -iq \gg K$  and get to leading order

$$\mathbf{B}_t[\mathbf{K}] \approx \frac{i\mu_0}{q} \mathbf{k}_t(\mathbf{n} \times \hat{\mathbf{K}}) \cdot \mathbf{J}[\mathbf{K}]. \quad (27)$$

Comparing to (24), we see that the skin effect effectively prevents the magnetic field from leaking out of the metal. It is sufficient to work to this order to get the large distance asymptotics [Eq. (26), second line].

### 3.4 Discussion

We start with a discussion of the change in the power laws (26), as one changes the distance from below the skin depth to much larger values. In the short distance regime, the effective volume inside the metal that contributes to the magnetic noise, is of the order  $z^3$ , since across the distance  $z$ , absorption is negligible. Adding incoherently magnetic fields with an amplitude  $\sim 1/z^2$  for each element in this volume, gives the  $1/z$  power law for the magnetic noise power. At larger distances, damping in the metal becomes relevant, and one expects only a surface layer of volume  $\sim z^2\delta$  to contribute. This leads to a scaling  $\delta/z^2$  that is not the one found here. In fact, as the skin depth gets shorter, the transmission through the metallic surface also becomes more inefficient, as discussed in Section 3.3. This leads to a reduced transmitted field. The calculation shows that in Fourier space, one factor  $1/K$  becomes  $1/q$  [Eqs. (13, 27)] so that the transmitted field scaling changes like  $1/z^2 \mapsto \delta/z^3$ . We get the  $\delta^3/z^4$  behaviour by incoherently summing this up over the near-surface volume  $\sim z^2\delta$ .

Let us compare to results obtained previously for metallic layers with finite thickness  $t$ . Calculations in

this geometry have been performed by Varpula and Poutanen [4], Sidles and co-workers [6], and Rekdal and co-workers [18]. At low frequencies where the skin depth becomes the largest scale, one gets

$$\delta \gg z, t: \quad \mathcal{B}_{ij}(\mathbf{r}; \omega) = \frac{\mu_0^2 \sigma k_B T}{16\pi} s_{ij} \frac{t}{z(z+t)}. \quad (28)$$

This is consistent with the simple rule of removing the vacuum half-space below the layer (replace  $1/z$  by  $1/z - 1/(z+t)$ ), ignoring the boundary conditions at the lower interface. One can actually show that sub-layer materials add negligible noise as long as its conductivity is much less than that of the metallic layer (see Refs. [18, 25]). In the limit of a thin layer,  $t \ll z$ , the noise is smaller compared to a half-space, because it decays like  $t/z^2$ . For cylindrical wires, the noise reduction is even stronger, see references [17, 21].

At higher frequencies, the skin depth becomes shorter, and different regimes emerge depending on the relative magnitude of distance  $z$  and thickness  $t$ . We focus in the following on the thin layer limit. Varpula and Poutanen [4] give the following empirical interpolation formula for a layer thinner than the skin depth

$$t \ll \delta, z: \quad \mathcal{B}_{ij}(\mathbf{r}; \omega) \approx \frac{\mu_0^2 \sigma k_B T}{16\pi} s_{ij} \frac{t/z^2}{1 + [4zt/(\pi^2\delta^2)]^2}. \quad (29)$$

This gives at large distance  $z \gg \delta$  a noise spectrum with a scaling  $\sim \delta^4/(tz^4)$ , similar to the half-space, but with an increased amplitude (by a factor of order  $\delta/t$ ). Note that in this regime, decreasing the layer thickness just produces the opposite effect on the noise. This unusual result has been confirmed experimentally in the kHz range [4]. A non-monotonic behaviour with either skin depth or conductivity  $\sigma = 2/(\mu_0\omega\delta^2)$  has also been pointed out by Rekdal and co-workers [17, 18]: bad conductors show only weak current fluctuations, while good conductors efficiently screen the magnetic field<sup>1</sup>. Equation (29) indeed yields a noise maximum for  $\delta \sim \sqrt{zt}$ , consistent with reference [18].

Finally, a layer thicker than the skin depth has been considered in references [6, 18], where the following asymptotics is derived

$$\mathcal{B}_{ij}(\mathbf{r}; \omega) \approx \frac{\mu_0^2 \sigma k_B T}{16\pi} s_{ij} \times \begin{cases} \frac{\delta^4}{2tz^4} & t \ll \delta \ll \sqrt{zt}, \\ \frac{3\delta^3}{2z^4} & \delta \ll \min(z, t). \end{cases} \quad (30)$$

Note that the first line differs from equation (29) by a numerical factor — this may be due to the chosen interpolation. The second line, consistent with reference [18], shows that for a very short skin depth, there is no difference between a metallic layer and a half-space [Eq. (26)].

<sup>1</sup> A similar situation occurs for the absorption of normally incident plane waves in a thin metallic film. The maximum absorption occurs for  $\delta \sim \sqrt{\lambda t}$ . See [6] for the link to magnetic noise and, e.g., [24] for an instructive discussion.

This could have been expected given the highly efficient screening.

Let us finally touch upon the case of a superconducting object. Sidles and co-workers [6] have argued that it suffices to use a complex conductivity and to make the replacement  $\sigma \mapsto \text{Re}[\sigma(\omega)]$ . For an ideal superconductor and zero temperature, London's equation yields  $\sigma(\omega) = -i\lambda_L^2/(\mu_0\omega)$  with  $\lambda_L$  the London penetration depth, and magnetic-near-field noise is completely suppressed. At finite temperature, the superconducting phase coexists with a normal phase, and  $\text{Re}[\sigma(\omega)]$  is finite. In terms of the (frequency-dependent) phase angle  $\varphi$  in  $\sigma = |\sigma|e^{-i\varphi}$ , the following interpolation formula is given in reference [6] for the magnetic noise spectrum above a superconducting layer<sup>2</sup>

$$\mathcal{B}_{ij}(\mathbf{r}; \omega) \approx \frac{\mu_0^2 k_B T |\sigma| \cos \varphi}{16\pi} s_{ij} \frac{3\delta^3 t}{\mathcal{D}} \quad (31)$$

$$\mathcal{D} = 3\delta^3 z(z+t) + 2\sqrt{2}(1 - e^{-\alpha_c}) \times tz^2(z + \delta\sqrt{2})^2 \cos(\pi/4 - \varphi/2) \quad (32)$$

$$\alpha_c = \frac{zt + 4\delta^2 \sin \varphi}{\sqrt{2} z \delta \cos(\pi/4 - \varphi/2)}. \quad (33)$$

Numerically, it is found that this formula reproduces the results of an exact calculation to within 2 dB. For a normal conductor ( $\varphi = 0$ ), it reproduces the asymptotics (26, 30).

Sidles and co-workers have also given corrections for a material with a weak magnetic susceptibility ( $|\mu - \mu_0| > 0$ ) where thermal magnetisation fluctuations contribute to the magnetic field noise as well (with a noise spectrum proportional to  $\text{Im} 1/\mu$ ), see equations (35, 36a) of reference [6]. Rekdal and co-workers [18] have pointed out that measurements of the magnetic susceptibility actually allow the inference the frequency-dependent complex conductivity that determines both the skin depth and magnetic noise properties. For niobium, the skin depth significantly differs from the London penetration depth at temperatures below the transition point and frequencies around 500 kHz.

## 4 Magnetic Green function

In this section, we switch to an alternative approach to magnetic-near-field noise that exploits a link to classical dipole radiation. In fact, the field radiated by a single point-like magnetic moment is sufficient to get the magnetic noise spectrum when the fluctuation-dissipation theorem (34) is used. This is a significant advantage for numerical calculations that are needed anyway in more complex geometries. In the 'incoherent summation' technique, one not only faces a similar effort to be invested in the computation of the field, but the calculation has to be repeated for a large number of nonequivalent sources (all volume elements filled with absorbing material). As pointed out by Sidles et al. [6], this redundancy can be

<sup>2</sup> We have corrected an obvious error in equation (6a) of reference [6] and took the classical limit  $\hbar\omega \ll k_B T$ .

avoided using the reciprocity theorem: once the field emitted by a suitable point source is computed, the relevant quantity is the total power absorbed in the metallic structure. In the following, we formulate the equations to solve near arbitrary metallic structures, with retardation in vacuum being neglected. Section 4.2 specialises to a half-space and shows that the reflected Green tensor is consistent with a fully retarded calculation.

### 4.1 Fluctuation-dissipation theorem

The fluctuation-dissipation (FD) theorem for the magnetic field reads [15]:

$$\mathcal{B}_{ij}(\mathbf{x}_1, \mathbf{x}_2; \omega) = 2\hbar f(\hbar\omega/k_B T) \text{Im} \mathcal{H}_{ij}(\mathbf{x}_1, \mathbf{x}_2; \omega) \quad (34)$$

where  $\mathcal{H}$  is the magnetic Green tensor, i.e., the magnetic field radiated by an oscillating magnetic point dipole  $\mathbf{m}$  at  $\mathbf{x}_2$ ,  $B_{\text{dip},i}(\mathbf{x}_1; \omega) = \sum_j \mathcal{H}_{ij}(\mathbf{x}_1, \mathbf{x}_2; \omega) m_j$ . For the low-frequency regime relevant here, the temperature dependence reduces to  $f(\hbar\omega/k_B T) = k_B T/\hbar\omega$ .

Since we are interested in atoms trapped in vacuum above a metallic structure, we shall take  $\mathbf{x}_1 = \mathbf{x}_2$  in vacuum. The magnetic dipole field  $\mathbf{B}_{\text{dip}}$  then can always be written as the sum of the vacuum radiation plus a field scattered or reflected from the structure. The vacuum field gives an imaginary part  $\text{Im} \mathcal{H}^{\text{vac}}(\mathbf{x}_1, \mathbf{x}_1; \omega)$  that reproduces Planck's formula for the blackbody radiation spectrum. We shall actually neglect this contribution compared to the one of the scattered field. For a planar surface, the scattered field can be written as an integral over Fresnel reflection coefficients (see, e.g., Ref. [16]). We check in the following section that the boundary conditions of Section 2 reproduce the Fresnel coefficients, at least in the low-frequency limit we focus on here.

We shall use the vector potential  $\mathbf{A}$  and the scalar potential  $\phi$  in the 'generalised Coulomb gauge'  $\nabla \cdot \varepsilon \mathbf{A} = 0$ . This gives the following wave equations for the domains outside and inside the metallic objects whose shape is left arbitrary for the moment. Outside the object:

$$\nabla^2 \phi = 0 \quad (35)$$

$$\nabla^2 \mathbf{A} = -\mu_0 \nabla \times \mathbf{M} - \frac{i\omega}{c^2} \nabla \phi \quad (36)$$

$$\nabla \cdot \mathbf{A} = 0 \quad (37)$$

where  $\mathbf{M}(\mathbf{x}) = \mathbf{m} \delta(\mathbf{x} - \mathbf{x}_1)$  is the magnetisation density for a point dipole. Inside the object:

$$\nabla^2 \phi = 0 \quad (38)$$

$$\nabla^2 \mathbf{A} + q^2 \mathbf{A} = \frac{q^2}{\omega} \nabla \phi \quad (39)$$

$$\nabla \cdot \mathbf{A} = 0. \quad (40)$$

We will consistently work in the limit  $|qc/\omega| \sim \lambda/\delta \rightarrow \infty$ , with spatial derivatives being comparable to  $q$ . We thus cover length scales comparable with or smaller than the skin depth. Combined with the boundary conditions for the potentials, these equations allow us to determine

the field everywhere. Note that there is no source term in the equations for the scalar potential  $\phi$ , even when the boundary conditions are taken into account. Without loss of generality, we therefore put  $\phi \equiv 0$  in the following.

## 4.2 Planar geometry

In the planar case, we have a simple analytical solution for the magnetic noise tensor — a benchmark result that has to be reproduced by our theory. Details of the calculation can be found in Appendix B. We use the boundary condition  $\mathbf{n} \cdot \mathbf{A}|_{\text{in}} = 0$  characteristic for the metal-vacuum interface [Eq. (5)] that follows from  $\mathbf{E} = i\omega\mathbf{A}$ . Translational symmetry allows us to expand the field radiated by the magnetic dipole in vacuum in plane waves with wave vector  $\mathbf{K}$  parallel to the interface. Focussing on an incident plane wave, the calculation yields a reflected magnetic field with Fourier amplitude

$$\mathbf{B}_r[\mathbf{K}] = \frac{K - \kappa}{K + \kappa} \mathbf{k}_r \left( \mathbf{n} \times \hat{\mathbf{K}} \right) \cdot \mathbf{A}_i[\mathbf{K}], \quad (41)$$

where  $\mathbf{A}_i[\mathbf{K}]$  is given in equation (53). The ratio  $(K - \kappa)/(K + \kappa)$  is the same reflection coefficient that appears in the (retarded) magnetic Green function, when the limit  $\lambda \rightarrow \infty$  is taken, see, e.g., [16]. In Appendix B, the magnetic Green tensor computed from (41) is found to be

$$\mathcal{B}_{ij}(\mathbf{r}; \omega) = -\frac{\mu_0 k_B T}{2\pi\omega} s_{ij} \int_0^\infty \frac{K^3 dK 2 \operatorname{Im} \kappa e^{-2Kz}}{|\kappa + K|^2}. \quad (42)$$

This expression agrees with equation (25) thanks to the identity  $2 \operatorname{Im} \kappa \operatorname{Re} \kappa = \operatorname{Im} \kappa^2 = -\operatorname{Im} q^2 = -\mu_0 \sigma \omega$ .

## 5 Summary and conclusion

We have discussed in this paper calculations for low-frequency magnetic noise fields at sub-wavelength distances to metallic objects. The role of the object surfaces has been clarified: they screen some field components so that only current elements parallel to the surface produce fields outside the object. This occurs even on a distance scale where dissipation in the metal is negligible. Neglecting this effect leads to errors up to a factor of three for the components of the magnetic noise tensor at short distance (smaller than the skin depth  $\delta$ ), and completely wrong power laws at larger distances ( $\gg \delta$ ). As a consequence, the simple incoherent addition of thermal noise fields has to be replaced by a more involved calculation, preferably based on the fluctuation-dissipation theorem for the field. We have formulated an outline of this calculation in a generic geometry, spelling out the boundary conditions that apply in the low-frequency regime characteristic for miniaturised atom traps. We hope that this opens a way to accurate and numerically efficient methods of characterizing magnetic noise spectra near complex atom chip structures.

Our theory can be extended to dielectric objects as well. If absorption is large and  $|\varepsilon| \gg 1$ , the same approach

can be carried over, with the skin depth defined by  $1/\delta = (\omega/c) \operatorname{Im} \sqrt{\varepsilon}$ . For a purely real permittivity, however, one has to include retardation in the vacuum regions to get a nonzero imaginary part in the magnetic Green function. When  $\varepsilon$  is of order unity, the boundary conditions for the fields assume, of course, their standard form. Numerical calculations are currently under way to test the validity of the non-retarded approach.

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## Appendix A: Transmission through a planar interface

We first compute the magnetic field radiated by a current distribution  $\mathbf{j}(\mathbf{x})$  localised inside a homogeneous metal. This is given by

$$\mathbf{B}_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla_{\mathbf{r}} \times \int dV(\mathbf{x}) \frac{e^{iq|\mathbf{r}-\mathbf{x}|}}{|\mathbf{r}-\mathbf{x}|} \mathbf{j}(\mathbf{x}) \quad (43)$$

where we recall that  $q^2 = 2i/\delta^2$ . To solve the transmission problem through a planar interface, it is expedient to use the expansion of the spherical wave in plane waves (the Weyl angular spectrum) with the wave vector  $\mathbf{K} = (k_x, k_y, 0)$ :

$$\frac{e^{iq|\mathbf{r}-\mathbf{x}|}}{4\pi|\mathbf{r}-\mathbf{x}|} = \frac{1}{2} \int \frac{d^2 K}{(2\pi)^2 \kappa} e^{i\mathbf{K} \cdot (\mathbf{r}-\mathbf{x}) - \kappa|z-x_3|}. \quad (44)$$

If we consider a point  $\mathbf{r}$  just below the interface, the absolute value in equation (44) can be dropped, and we get a Fourier coefficient

$$\mathbf{B}_i[\mathbf{K}] = \frac{i\mu_0}{2\kappa} \mathbf{k}_i \times \mathbf{J}[\mathbf{K}] \quad (45)$$

where  $\mathbf{k}_i = \mathbf{K} + i\kappa\mathbf{n}$ , and  $\mathbf{J}[\mathbf{K}]$  is given by equation (11), with  $K$  in the exponential replaced by  $\kappa$ . Only in the quasi-static limit,  $\kappa \rightarrow K$ , and we recover equation (10).

In the same way, we get the normal derivative

$$\frac{\partial}{\partial n} \mathbf{B}_i[\mathbf{K}] = -\kappa \mathbf{B}_i[\mathbf{K}] = -\frac{i\mu_0}{2} \mathbf{k}_i \times \mathbf{J}[\mathbf{K}]. \quad (46)$$

The first equation reflects the property that the magnetic field created by the current is propagating (in fact, decaying) in the upward direction. For the reflected and transmitted fields, similar relations hold:

$$\frac{\partial}{\partial n} \mathbf{B}_r[\mathbf{K}] = +\kappa \mathbf{B}_r[\mathbf{K}], \quad \frac{\partial}{\partial n} \mathbf{B}_t[\mathbf{K}] = -K \mathbf{B}_t[\mathbf{K}]. \quad (47)$$

The jump condition (8) thus gives

$$-K \mathbf{B}_t[\mathbf{K}] + \kappa (\mathbf{B}_i[\mathbf{K}] - \mathbf{B}_r[\mathbf{K}]) = \mu_0 \sigma \mathbf{n} \times \mathbf{E}_t[\mathbf{K}]. \quad (48)$$

It turns out that to proceed, we do not actually need to compute the electric field at the metallic surface. We combine the continuity of the magnetic field perpendicular to the surface with the corresponding component of equation (48) and solve for the transmitted field component

$$\mathbf{n} \cdot \mathbf{B}_t[\mathbf{K}] = \frac{2\kappa}{\kappa + K} \mathbf{n} \cdot \mathbf{B}_i[\mathbf{K}] \quad (49)$$

where we recognise one of the Fresnel coefficients. The component along  $\mathbf{K}$  follows from  $\text{div } \mathbf{B}_t = \mathbf{i}\mathbf{k}_t \cdot \mathbf{B}_t[\mathbf{K}] = 0$ .

The last component is along  $\mathbf{n} \times \mathbf{K}$ , and we can use the following trick to show that it actually vanishes. The Maxwell equations yield ( $\alpha = i, r, t$ )

$$i(\mathbf{n} \times \mathbf{K}) \cdot \mathbf{B}_\alpha[\mathbf{K}] = \mathbf{i}\mathbf{n} \cdot (\mathbf{k}_\alpha \times \mathbf{B}_\alpha[\mathbf{K}]) = \mu_0 \sigma_\alpha \mathbf{n} \cdot \mathbf{E}_\alpha[\mathbf{K}] \quad (50)$$

with  $\sigma_{i,r} = \sigma$  and  $\sigma_t \rightarrow 0$ . We now use the boundary condition that the normal electric field is zero inside the metal [Eq. (5)], and get from equation (50)  $(\mathbf{n} \times \mathbf{K}) \cdot (\mathbf{B}_i[\mathbf{K}] + \mathbf{B}_r[\mathbf{K}]) = 0$ . Now, this is a magnetic field component tangential to the surface, and therefore

$$(\mathbf{n} \times \mathbf{K}) \cdot \mathbf{B}_t[\mathbf{K}] = 0. \quad (51)$$

Combining the two nonzero components found above, we readily get the transmitted field (24).

## Appendix B: Magnetic Green tensor

The vector potential created by a magnetic point dipole in free space (solution of Eq. (36)) is of the form

$$\mathbf{A}_i(\mathbf{r}) = -\frac{\mu_0}{4\pi} \nabla_{\mathbf{r}} \times \frac{\mathbf{m}}{|\mathbf{r} - \mathbf{x}_1|} \quad (52)$$

where  $\mathbf{x}_1$  is the dipole position. Above a planar surface, we use the Weyl expansion (44) and find the following Fourier coefficient for the field incident on the interface:

$$\mathbf{A}_i[\mathbf{K}] = -\frac{i\mu_0}{2K} \mathbf{k}_i \times \mathbf{m} e^{-i\mathbf{k}_i \cdot \mathbf{x}_1} \quad (53)$$

with an incident wavevector  $\mathbf{k}_i = \mathbf{K} - iK\mathbf{n}$ . The reflected field is characterised by a wavevector  $\mathbf{k}_r = \mathbf{K} + iK\mathbf{n}$  and transversal as well, i.e.  $\mathbf{k}_r \cdot \mathbf{A}_r[\mathbf{K}] = 0$ .

The transmitted field has the wavevector  $\mathbf{k}_t = \mathbf{K} - i\kappa\mathbf{n}$  with  $K^2 - \kappa^2 = q^2$  ( $\text{Re } \kappa > 0$ ). Here, we have two ‘transversality conditions’:

$$\mathbf{k}_t \cdot \mathbf{A}_t[\mathbf{K}] = 0, \quad \mathbf{n} \cdot \mathbf{A}_t[\mathbf{K}] = 0, \quad (54)$$

where the second one actually comes from the boundary condition for the electric field, equation (5). We conclude that the transmitted field is parallel to the vector  $\mathbf{n} \times \mathbf{k}_t = \mathbf{n} \times \mathbf{K}$ . This vector lies inside the boundary and is perpendicular to  $\mathbf{K}$ . Since the tangential components of the vector potential are continuous, the reflected field  $\mathbf{A}_r$  has to cancel the component of  $\mathbf{A}_i$  parallel to  $\mathbf{K}$ . This gives a first condition for the reflected field:

$$\mathbf{K} \cdot (\mathbf{A}_i[\mathbf{K}] + \mathbf{A}_r[\mathbf{K}]) = 0. \quad (55)$$

We need a second boundary condition to solve the problem. This is, of course, the continuity of the magnetic field  $\mathbf{B} = \nabla \times \mathbf{A}$ . The magnetic field on the inner side of the interface is computed to be

$$\mathbf{B}|_{\text{in}} = \mathbf{i}\mathbf{k}_t \times \mathbf{A}_t = i(\mathbf{n}K^2 + i\kappa\mathbf{K}) t \quad (56)$$

where  $t$  is an un-normalised transmission coefficient. Computing the normal and  $\mathbf{K}$  component on the ‘outer side’, we get the linear system

$$K^2 t = -\mathbf{i}\mathbf{n} \cdot \mathbf{B}|_{\text{out}} = (\mathbf{n} \times \mathbf{K}) \cdot (\mathbf{A}_i + \mathbf{A}_r) \quad (57)$$

$$i\kappa K^2 t = -i\mathbf{K} \cdot \mathbf{B}|_{\text{out}} = iK(\mathbf{n} \times \mathbf{K}) \cdot (\mathbf{A}_i - \mathbf{A}_r) \quad (58)$$

whose solution involve the standard Fresnel reflection and transmission coefficients:

$$t = \frac{2}{K(K + \kappa)} (\mathbf{n} \times \mathbf{K}) \cdot \mathbf{A}_i, \quad (59)$$

$$(\mathbf{n} \times \mathbf{K}) \cdot \mathbf{A}_r = \frac{K - \kappa}{K + \kappa} (\mathbf{n} \times \mathbf{K}) \cdot \mathbf{A}_i. \quad (60)$$

This yields the following expression for the reflected field

$$\mathbf{A}_r = -\frac{\mathbf{k}_r}{K} (\hat{\mathbf{K}} \cdot \mathbf{A}_i) + \frac{K - \kappa}{K + \kappa} (\mathbf{n} \times \hat{\mathbf{K}}) (\mathbf{n} \times \hat{\mathbf{K}}) \cdot \mathbf{A}_i \quad (61)$$

whose first (‘longitudinal’) term ensures condition (55) while still being ‘transversal’ (this is due to the fact that  $\mathbf{k}_r^2 = 0$  in the limit  $\lambda \rightarrow \infty$ ). The corresponding magnetic field is determined by the second term only and one finds equation (41).

From equation (61), we identify the following magnetic Green tensor

$$\mathcal{H}(\mathbf{r}, \mathbf{r}) = \mathcal{H}^{(\text{vac})}(\mathbf{r}, \mathbf{r}) + \mathcal{H}^{(\text{ref})}(\mathbf{r}, \mathbf{r}) \quad (62)$$

$$\mathcal{H}^{(\text{ref})}(\mathbf{r}, \mathbf{r}) = \int \frac{d^2 K}{(2\pi)^2} \frac{\mu_0 e^{-2Kz}}{2K} \frac{K - \kappa}{K + \kappa} \mathbf{k}_r \otimes \mathbf{k}_i \quad (63)$$

using the identity  $(\mathbf{n} \times \hat{\mathbf{K}}) \times \mathbf{k}_i = \mathbf{i}\mathbf{k}_i$ . The integration over the azimuthal angle amounts to angular averaging:  $\langle \mathbf{k}_r \otimes \mathbf{k}_i \rangle = K^2 s_{ij}$ . For the fluctuation-dissipation theorem, we need the imaginary part of the reflection coefficient

$$\text{Im} \frac{K - \kappa}{K + \kappa} = \frac{-2K \text{Im } \kappa}{|K + \kappa|^2}. \quad (64)$$

The vacuum Green tensor is purely real in the static limit  $\lambda \rightarrow \infty$  we focus on here, and does not contribute to the magnetic noise spectrum (34). Putting everything together, we get the Green tensor (42).

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